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AXISYMMETRIC SELF-SIMILAR EQUILIBRIA OF SELF-GRAVITATING ISOTHERMAL SYSTEMS

Mikhail V. Medvedev[1] and Ramesh Narayan
Harvard-Smithsonian Center for Astrophysics, 60 Garden St., Cambridge, MA 02138

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ABSTRACT

All axisymmetric self-similar equilibria of self-gravitating, rotating, isothermal systems are identified by solving the nonlinear Poisson equation analytically. There are two families of equilibria: (1) Cylindrically symmetric solutions in which the density varies with cylindrical radius as $R^{-\alpha}$, with $0 \le \alpha \le 2$. (2) Axially symmetric solutions in which the density varies as $f(\theta)/r^2$, where r is the spherical radius and θ is the co-latitude. The singular isothermal sphere is a special case of the latter class with $f(\theta) =$ constant. The axially symmetric equilibrium configurations form a two-parameter family of solutions and include equilibria which are surprisingly asymmetric with respect to the equatorial plane. The asymmetric equilibria are, however, not force-free at the singular points $r = 0, \infty$, and their relevance to real systems is unclear. For each hydrodynamic equilibrium, we determine the phase-space distribution of the collisionless analog.

Subject headings: galaxies: structure — galaxies: kinematics and dynamics — stars: kinematics — stars: formation — ISM: kinematics and dynamics — hydrodynamics

1. INTRODUCTION

Many astrophysical objects consist of equilibrium configurations in which self-gravity is resisted by pressure (velocity dispersion) and centrifugal forces. A particularly simple example, which is well-suited for practical analysis, is the case of an isothermal fluid with a linear pressure–density relation, $p = \rho c_s^2$, where p is the pressure, ρ is the density, and c_s is the sound speed which is taken to be a constant. Equilibrium configurations of isothermal systems have been studied as models of galaxies (Toomre 1982; Binney & Tremaine 1987; see also Richstone 1980; Monet et al. 1981 for some special cases) and newly formed stars (e.g., Hayashi et al. 1982; Kiguchi et al. 1987).

Although some analytical solutions have been published previously, no systematic study of isothermal equilibria has been presented so far. In this paper we classify and derive analytically all possible self-similar axisymmetric equilibria of a self-gravitating isothermal system. We find that there are only two distinct classes of equilibria:

1. Cylindrically symmetric equilibria, in which all quantities, such as density, potential, velocity, etc., depend on the cylindrical radius, R, only;

2. Axially symmetric equilibria, in which the quantities are functions of both the spherical radius, r, and the co-latitude, θ.

The cylindrically symmetric solutions are rather simple: the density varies as $\rho \propto R^{-2(n+1)}$, and the azimuthal velocity as $\Omega R \equiv v_\varphi = v_{\varphi 0} R^{-n}$, where the allowed range of n is $-1 \le n \le 0$.

The axially symmetric solutions are more rich. First, the density always varies as $\rho \propto 1/r^2$ and the rotation curve is flat, $v_\varphi = v_{\varphi 0} =$ constant. Second, these equilibria form a two-parameter family of solutions. One of the parameters, A, determines the rotation velocity, $A = 2 + v_{\varphi 0}^2/c_s^2$, and the other parameter, B, controls the symmetry of the solutions with respect to the equatorial plane. For $B = 1$, the solutions are up-down symmetric. These solutions have been previously obtained by Toomre (1982) and Hayashi et al. (1982) and include the singular isothermal sphere ($A = 2$) and the cold Mestel disk ($A \to \infty$) as limiting cases.

The solutions with $B \ne 1$ are asymmetric with respect to the equatorial plane. This contradicts Lichtenstein's theorem (Lichtenstein 1933; Wavre 1932) which states that a barotropic, self-gravitating equilibrium must have a plane of symmetry perpendicular to the axis of rotation. The paradox is resolved by noting that the solutions with $B \ne 1$ are not force-free at two singular points, $r = 0$ and $r = \infty$, where Poisson's equation is ill-defined. External forces have to be applied to the matter at the singularities to hold the system in equilibrium. This is proved rigorously for the case without rotation, $A = 2$, and is likely to be correct also for rotating solutions.

The paper is organized as follows. In §2, we present a general analysis of the problem and classify all possible self-similar equilibrium configurations of self-gravitating isothermal systems. We derive cylindrically symmetric solutions of the equation of hydrostatic equilibrium in §3. We identify a two-parameter family of axially symmetric solutions in §4 and describe the properties of these solutions in §5. In §6 we discuss the non-rotating case and in §7 the thin disk limit. In §8 we determine the steady-state distribution function of collisionless stellar systems, and we summarize the conclusions in §9.

2. GENERAL CONSIDERATIONS AND SELF-SIMILAR SOLUTIONS

We perform all calculations in spherical coordinates (r, θ, φ), but we also occasionally consider the cylindrical radius, $R = r \sin\theta$.

[1] Also at the Institute for Nuclear Fusion, RRC "Kurchatov Institute", Moscow 123182, Russia; E-mail: mmedvedev@cfa.harvard.edu; URL: http://cfa-www.harvard.edu/~mmedvede/

A hydrostatic equilibrium configuration satisfies the momentum equation with vanishing time derivative, and Poisson's equation. By definition, the $\hat{r}$- and $\hat{\theta}$-components ("hat" denotes unit vectors) of the velocity vanish in equilibrium. Also, by the condition of axial symmetry, all derivatives with respect to the toroidal angle, φ, vanish. Thus, we need to consider only the $\hat{r}$- and $\hat{\theta}$-components of the momentum equation and Poisson's equation for the potential:

$$\frac{v_\varphi^2}{r} = \frac{c_s^2}{\rho}\frac{\partial \rho}{\partial r} + \frac{\partial \phi}{\partial r}, \tag{1a}$$

$$\frac{v_\varphi^2 \cot\theta}{r} = \frac{c_s^2}{\rho r}\frac{\partial \rho}{\partial \theta} + \frac{1}{r}\frac{\partial \phi}{\partial \theta}, \tag{1b}$$

$$\nabla^2_{r,\theta}\, \phi = 4\pi G \rho, \tag{1c}$$

where

$$\nabla^2_{r,\theta} = \frac{1}{r^2}\frac{\partial}{\partial r}\, r^2 \frac{\partial}{\partial r} + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\, \sin\theta \frac{\partial}{\partial \theta}$$

is the axisymmetric Laplacian operator, ϕ is the gravitational potential, v_φ is the toroidal component of the velocity due to rotation which is, in general, a function of r and θ, and G is Newton's constant. The radial coordinate and the density are taken to be dimensionless throughout the paper.

2.1. *Equilibrium of an Isothermal Gas*

We begin with a general treatment of the problem, without any assumption of self-similarity. We take the gas to be isothermal, i.e., the pressure and the density are related as follows,

$$p = c_s^2 \rho, \tag{2}$$

where $c_s =$ const. is the sound speed. Since an isothermal system is a special case of a barotropic system, $p = p(\rho)$, the following generic properties immediately follow from the Poincaré-Wavre theorem (Tassoul 1978): (i) the angular velocity is constant on cylinders centered on the axis of rotation, i.e., in cylindrical coordinates (R, φ, z), the velocity $v_\varphi = v_\varphi(R)$ is independent of z; (ii) the effective gravity can be derived from an effective potential; and (iii) the effective gravity is normal to the isopycnic (i.e., constant density) surfaces. Tassoul (1978) has derived general relations between ρ, ϕ, and v_φ in a rotating barotropic system; we briefly re-derive some of these results for completeness.

Eliminating v_φ between Eqs. (1a,b), we arrive at the following differential equation:

$$\left(\frac{\partial}{\partial \ln r} - \frac{\partial}{\partial \ln \sin\theta}\right)\left(c_s^2 \ln\rho + \phi\right) = 0. \tag{3}$$

Any function of the argument $\ln(r\sin\theta)$ is a solution of this differential equation. Absorbing the logarithm into the definition of the function, we write the solution as follows

$$\phi(r,\theta) = -c_s^2 \ln\rho(r,\theta) + u(r\sin\theta), \tag{4}$$

where u is an arbitrary function of the cylindrical radius $R = r\sin\theta$. The function u is related to the rotation velocity as follows,

$$v_\varphi^2(r,\theta) = v_\varphi^2(R) = R\,\frac{\partial\, u(R)}{\partial\, R}. \tag{5}$$

This relation is obtained upon substituting Eq. (4) into Eq. (1a).

Next, we use Poisson's equation to relate the density and gravitational potential. Substituting Eq. (4) into Eq. (1c) and using Eq. (5) and the identity

$$\nabla^2_{r,\theta}\, u(r\sin\theta) = \nabla^2_R\, u(R) = \frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial}{\partial R}\, u(R)\right),$$

we obtain the following equation for the density distribution,

$$-c_s^2 \nabla^2_{r,\theta}\, \ln\rho + \frac{1}{R}\frac{\partial}{\partial R}\, v_\varphi^2 = 4\pi G\rho. \tag{6}$$

Given the rotation curve, $v_\varphi(R)$, the solution of Eq. (6), together with Eqs. (2),(4), and (5), completely determines the solution. To proceed further, we need to make some simplifying assumptions.

2.2. *Self-Similar Solutions*

We now look for self-similar solutions of Eq. (6). Let us assume that v_φ is described by a power-law in $R = r\sin\theta$. Then the density distribution is also a power-law in r, and we write

$$\rho = \rho_0 \frac{f(\theta)}{r^\alpha}, \qquad v_\varphi = \frac{v_{\varphi 0}}{R^n}, \tag{7}$$

where $f(\theta)$ is an unknown function to be calculated, ρ_0 and $v_{\varphi 0}$ are normalization constants, and α and n are parameters. Substituting Eqs. (7) into Eq. (6), we obtain

$$\frac{c_s^2}{r^2}\left(\alpha - \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\, \sin\theta\frac{\partial}{\partial\theta}\, \ln f(\theta)\right) - \frac{2n\, v_{\varphi 0}^2}{(r\sin\theta)^{2n+2}} = 4\pi G\rho_0 \frac{f(\theta)}{r^\alpha} \tag{8}$$

This equation can be satisfied only when the powers of r on the various terms match. There are two possible cases:

1. Cylindrically symmetric solutions. In this case, the first term on the left-hand-side of Eq. (8) vanishes identically and $2n + 2 = \alpha$.

2. Axially symmetric solutions. In this case, $\alpha = 2$ and $n = 0$ and the term proportional to $v_{\varphi 0}$ in Eq. (8) vanishes. These solutions have flat rotation curves.

The above list exhausts all possible cases. Thus, there are only two families of self-similar, axisymmetric equilibrium configurations of self-gravitating isothermal systems with permanent rotation.

We now describe the two families of solutions in detail. In §3 we discuss the cylindrical solutions, and in §§4–7 we discuss the axially symmetric solutions.

3. CYLINDRICAL SOLUTIONS

In this case, the first term on the left-hand-side of Eq. (8) vanishes identically, so that $2n+2 = \alpha$. This condition forces f to be $f(\theta) = (\sin\theta)^{-\alpha}$. The self-similar solution for ρ in terms of $R = r\sin\theta$ is easily obtained from Eq. (8),

$$\rho(R) = -\frac{n\, v_{\varphi 0}^2}{2\pi G}\frac{1}{R^{2n+2}}. \tag{9a}$$

We notice that this solution is physical only for $n \le 0$; otherwise, the density is negative. The gravitational potential is determined from Eq. (4). We have,

$$\phi = -c_s^2 \ln\left(-\frac{n\, v_{\varphi 0}^2}{2\pi G}\frac{1}{R^{2n+2}}\right) + u(R) + \phi_0, \tag{9b}$$

where ϕ_0 is a constant that defines the zero-level of the potential and the function $u(R)$ is obtained by integrating Eq. (5) for a given power-law rotation curve,

$$u(R) = \begin{cases} v_{\varphi 0}^2/(2n+2)R^{2n+2}, & \text{if } n \neq 0, \\ v_{\varphi 0}^2 \ln(R), & \text{if } n = 0. \end{cases}$$

The solution is well behaved everywhere except at the axial singularity,[2] where a Dirac δ-function, i.e., a central mass "wire," may be located. This additional mass density proportional to $\delta(R)$ may be calculated using Gauss' integral theorem for the flux of the gravitational field through a surface,

$$\int_S (-\nabla\phi)\cdot \mathbf{n}\, \mathrm{d}s = -4\pi G M_s, \tag{10}$$

where M_s is the mass enclosed within a volume bounded by a surface S and $\mathbf{n}$ is the unit normal outward from the volume. We choose the surface S to be an axial cylinder of radius ΔR and length L, centered at $R = 0$. Since ϕ is a function of R, [cf., Eq. (9b)], the gradient $\nabla\phi$ is radial and is equal to

$$\nabla\phi = \hat{R}\left[\frac{(2n+2)c_s^2}{\Delta R} + \frac{v_{\varphi 0}^2}{\Delta R^{2n+1}}\right], \tag{11}$$

where $\hat{R}$ is the unit vector along the cylindrical radius. Calculating the enclosed mass via Gauss' theorem and taking the limit $\Delta R \to 0$ (remember, $n \le 0$), we obtain the following expression for the linear density of the "central wire,"

$$\mu_s \equiv \frac{\partial M_s}{\partial L} = \begin{cases} (1-|n|)\, c_s^2/G, & \text{if } n < 0, \\ (c_s^2 + v_{\varphi 0}^2/2)/G, & \text{if } n = 0. \end{cases} \tag{12}$$

Clearly, the mass at the singularity becomes negative for $|n| > 1$. Thus, the self-similar, cylindrically symmetric solutions exist only for n in the range $-1 \le n \le 0$. The magnitude of the rotation velocity $v_{\varphi 0}$ is arbitrary. It is worthwhile to note that for $n = -1$, the axial δ-function singularity disappears ($\mu_s = 0$) and the solution, Eqs. (9), is well behaved everywhere. This case corresponds to solid-body rotation, $\Omega(R) = v_\varphi/R = \text{constant}$.

4. AXIALLY SYMMETRIC SOLUTIONS

The equation for ρ [Eq. (6) with $\alpha = 2$ and $n = 0$] reads

$$2 - \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\ln f(\theta)\right) = \frac{4\pi G\rho_0}{c_s^2} f(\theta). \tag{13}$$

As one can see, the velocity gradient term drops out. That is, the density distribution appears to be independent of the rate of rotation, in apparent contradiction to the Poincaré-Wavre theorem. As we shall see from the careful analysis below, this is not the case.

4.1. *Regular Solution*

It is remarkable that the nonlinear, second-order differential equation (13) may be solved analytically and a general two-parameter family of self-similar solutions can be found in a closed and explicit form. All details of this computation are presented in Appendix A. The solution is

$$\rho(r,\theta) = \frac{c_s^2}{2\pi G}\frac{A^2}{(r\sin\theta)^2}\frac{B\tan^A(\theta/2)}{\left[1 + B\tan^A(\theta/2)\right]^2}, \tag{14a}$$

where $A > 0$ and $B > 0$ are two free parameters. We have restricted A to be positive, because a change of sign of A is identical to the replacement: $B \to 1/B$. The gravitational potential calculated from Eq. (4) may be written in the form

$$\phi(r,\theta) = -c_s^2 \ln\left(\frac{\rho(r,\theta)}{|r\sin\theta|^{v_{\varphi 0}^2/c_s^2}}\right) + \phi_0. \tag{14b}$$

The denominator in the logarithm is the contribution due to rotation, $u(r\sin\theta) = v_{\varphi 0}^2 \ln(r\sin\theta) + \text{const.}$, as follows from Eq. (5). The requirement of regularity of the solution, i.e., the continuity of ρ and ϕ and their derivatives everywhere, constrains one of the free parameters,

$$A = 2 + v_{\varphi 0}^2/c_s^2 \ge 2 \tag{15}$$

(see §4.2 for more details). The other free parameter, B, remains unconstrained. Note that the density distribution depends on the rotation velocity through Eq. (15), in agreement with the Poincaé-Wavre theorem. For the special case $B = 1$, the solution (14a) reduces to the solution obtained by Toomre (1982) and Hayashi et al. (1982).[3]

4.2. *Axial Singularity and a Generalized Solution*

The solution (14) is smooth and well-behaved in the domain $0 < \theta < \pi$. However, the differential operator in Eq. (13) is ill-defined at $\theta = 0$. Therefore, a singular solution, proportional to the Dirac delta-function, $\delta(\theta)$, is allowed on the axis. This additional mass density may be calculated using Gauss' theorem, Eq. (10). As a surface of integration, we choose a segment of a cone of revolution centered on the axis consisting of the following pieces,

$$S_1 = \{r = r_0;\ 0 \le \theta \le \Delta\theta\},$$
$$S_2 = \{r_0 \le r \le r_0 + \Delta r;\ \theta = \Delta\theta\},$$
$$S_3 = \{r = r_0 + \Delta r;\ 0 \le \theta \le \Delta\theta\},$$

where r_0, Δr, and $\Delta\theta$ are constants. The unit normals to the three surface pieces are $\mathbf{n}_1 = -\hat{r}$, $\mathbf{n}_2 = \hat{\theta}$, and $\mathbf{n}_3 = \hat{r}$, respectively. The gradient of the potential for small angles reads

$$\nabla\phi \simeq \left(\frac{\beta c_s^2}{r};\ \frac{(\beta - A)c_s^2}{r\theta};\ 0\right), \tag{16}$$

[2] The singularity of the solution at $\theta = 0$ is due to the singularity of the differential operator of Eq. (8) on the axis.

[3] The parameter $A \equiv 2n + 2$ in Toomre (1982) and $A \equiv 2\gamma$ in Hayashi et al. (1982).

where $\beta = 2 + v_{\varphi 0}^2/c_s^2$. Performing the integration over the surface $S = S_1 \oplus S_2 \oplus S_3$, taking the limit $\Delta\theta \to 0$, and noticing that the radial distance, Δr, at $\theta = 0$ is the length along the axis, $\Delta r|_{\theta=0} = L$, we obtain the linear mass density of a singular density distribution on the axis,

$$\mu_s \equiv \frac{\partial M_s}{\partial L} = \frac{c_s^2}{2G}\left(2 + \frac{v_{\varphi 0}^2}{c_s^2} - A\right). \tag{17}$$

The requirement of a non-singular density, $\mu_s = 0$, gives back the relation (15). In the more general case, we see that A is related to the rotation velocity and the mass density at the axis. The mass on the axis cannot be negative, hence we have the constraint

$$A \le 2 + v_{\varphi 0}^2/c_s^2. \tag{18}$$

We note that the radial part of the spherical Laplacian, $\nabla_{r,\theta}^2$, is ill-defined at $r = 0$, so that the solution may also contain a point mass, $\delta(r)$, at the origin. However, using Gauss' theorem for a sphere of radius $\Delta r \to 0$ enclosing the origin, we find no singular mass can be "hidden" at $r = 0$.

In the rest of the paper we consider only the regular solutions with $\mu_s = 0$. These solutions satisfy $A = 2 + v_{\varphi 0}^2/c_s^2$ [equation (15)].

5. PROPERTIES OF THE AXIALLY SYMMETRIC SOLUTIONS

5.1. *General Properties*

Figures 1—4 show typical examples of the self-similar solution (14). Contours of equal density for different values of the parameters A and B are displayed. As is seen, the parameter A determines the overall shape of the matter distribution. As $A \to 0$ the configuration tends to the cylindrically symmetric limit, and an axial singularity with $\mu_s \neq 0$ is always present [see Eq. (17)]. Figure 1 shows an example with $A = 0.7$. The density does not vanish at infinity at $\theta = 0,\ \pi$. Solutions without an axial singularity, $\mu_s = 0$, exist for $A \ge 2$, cf., Eq. (15). The non-rotating limit, $A = 2$, yields confocal ellipsoids/spheres, as shown in Fig. 2 (see §6). For $A > 2$, toroidal configurations with rotation are obtained, as shown in Figs. 3 and 4. Interestingly, as $A \to \infty$, the profile flattens and tends to a thin disk configuration, as shown in Fig. 4 for $A = 20$.

The parameter B is responsible for up-down asymmetry. For $B = 1$, the solutions are symmetric with respect to the equatorial plane. The solutions with $B > 1$ are shifted (distorted) upwards, those with $B < 1$ are shifted downwards. Note that solutions with B and $1/B$ are identical except that they are turned upside-down with respect to each other. Note also that the solutions with $B \neq 1$ violate Lichtenstein's theorem (Lichtenstein 1933; Wavre 1932), which proves the existence of an equatorial plane of symmetry. The resolution of this paradox is discussed below (§6).

5.2. *The Finite System Limit*

We now determine some global properties, such as the total mass, M, gravitational energy, W, and angular momentum, $\mathbf{L}$, of the axially symmetric self-similar solutions as functions of A and B. These quantities diverge unless a cutoff is imposed at large radii. The most natural way to do this is to assume that the system is immersed in a medium with finite pressure, $p_{\rm ext}$, but vanishing density. As follows from Eq. (2), the cutoff surface in this case is the iso-density surface with $\rho_c = p_{\rm ext}/c_s^2$. The equation for the cutoff surface is

$$r_c(\theta) = \sqrt{g\,\Theta(\theta)/\rho_c}, \tag{19}$$

where $g = c_s^2/(2\pi G)$ and

$$\Theta(\theta) = \frac{A^2}{\sin^2\theta}\,\frac{B\tan^A(\theta/2)}{\left[1 + B\tan^A(\theta/2)\right]^2}$$

is the angular part of the function, $\rho(r,\theta)$. Notice that if $\mu_s \neq 0$, M and W still diverge due to the infinite mass located on the axis. So, we set $\mu_s = 0$, which means that we are restricted to the solutions with $A \ge 2$. The total mass, gravitational potential energy, and angular momentum are defined as follows,

$$M = \int_V \rho\,\mathrm{d}V, \quad W = \frac{1}{2}\int_V \rho\,\phi\,\mathrm{d}V, \quad \mathbf{L} = \int_V \rho\,\mathbf{v}\times\mathbf{r}\,\mathrm{d}V. \tag{20}$$

The binding energy is then $E = -(W + K)$, where the kinetic energy is $K = Mv_{\varphi 0}^2/2$. The surfaces of constant density and constant potential do not coincide due to rotation, unless $A = 2$. For this reason, the potential at the cutoff surface is a function of position, $\phi_c = \phi\,(r_c(\theta), \theta)$. To get rid of insignificant constants in the expression for W, we redefine the gravitational potential such that the maximum potential at the surface is zero. This constrains the constant ϕ_0. The value of $\theta_{\rm m}$, the extremum point of ϕ along the $r_c(\theta)$, is found to satisfy the equation, $\tan^A(\theta_{\rm m}/2) = 1/B$. Then, we have

$$\phi_0 = c_s^2\ln\rho_c - v_{\varphi 0}^2\ln\sqrt{g/\rho_c} - v_{\varphi 0}^2\ln(A/2). \tag{21}$$

As shown in §6, the solutions with $B \neq 1$ are not force-free; they include an external gravitational potential gradient. We do not include the external potential in our definition of the gravitational energy, W. We now calculate all the quantities (note, L_z is the only nonvanishing component of the angular momentum). The details of the calculation are given in Appendix B. The results are

$$M = \frac{2\pi g^{3/2}}{\sqrt{\rho_c}}\,\frac{\pi(A^2-4)}{16\cos(\pi/A)}\left(B^{1/A} + B^{-1/A}\right), \tag{22a}$$

$$W = -\frac{\pi g^{3/2}c_s^2}{\sqrt{\rho_c}}\,\frac{\pi(A-2)^2}{16\cos(\pi/A)}\Bigg(2 + (A+2)\Bigg[\frac{1}{2} - \ln 2 - \gamma + \frac{\pi}{2}\tan\frac{\pi}{A} - \psi\left(\frac{1}{2} + \frac{1}{A}\right)\Bigg]\Bigg)\left(B^{1/A} + B^{-1/A}\right), \tag{22b}$$

$$L_z = \frac{\pi g^2 c_s}{\rho_c}\,\frac{\pi(A^2-1)}{12\sin(\pi/A)}\sqrt{A-2}\left(B^{1/A} + B^{-1/A}\right), \tag{22c}$$

where $g = c_s^2/(2\pi G)$, $\psi(x) = \Gamma'(x)/\Gamma(x)$ is the digamma function, and $\gamma \equiv -\psi(1) = 0.57721\ldots$ is Euler's constant. The dependences of M, W and L_z as functions of A are plotted in Fig. 5. All these increase with increasing A, i.e., as the rotation velocity $v_{\varphi 0}$ increases. The dependence of these quantities as a function of B is trivial; all are proportional to $\left(B^{1/A} + B^{-1/A}\right)$ and have minima at $B = 1$, the symmetric configuration.

We emphasize that the truncation of a system at a finite $\rho = \rho_c$ breaks the self-similarity of the solutions. Thus, the above equations are approximate and should be used with care. Nevertheless, Eqs. (22) suggest that both the specific angular momentum of a finite configuration, L_z/M, and the gravitational binding energy per unit mass, W/M, are functions only of A and independent of B. The kinetic energy per unit mass, $v_{\varphi 0}^2/2$, is also a function only of A. Finally, if we consider the specific entropy, $s = S/M$, where $S \propto -\int_V \rho \ln \rho \, \mathrm{d}V$, we see that S differs from the gravitational energy by additive and multiplicative constants only. Thus s is again independent of B. The question then is: if energy and entropy are indistinguishable and independent of B, what determines which value of B is selected by a finite isothermal system with a given value of L_z/M? The answer, as we show in the next section, is the boundary condition, namely, the magnitude of external forces acting on the mass.

6. THE NONROTATING LIMIT, $A = 2$

We first prove that nonrotating solutions, $A = 2$, consist of nested confocal ellipsoids. The solution (14a) reads

$$\rho(r,\theta) = \frac{2c_s^2}{\pi G} \frac{1}{(r\sin\theta)^2} \frac{B\tan^2(\theta/2)}{\left[1 + B\tan^2(\theta/2)\right]^2}. \quad (23)$$

The equation of an iso-density contour line is obtained by setting $\rho(r,\theta) = \rho_1 =$ constant. Upon straightforward trigonometric simplifications, we obtain

$$\begin{aligned} r(\theta) &= \left(\frac{2c_s^2 B}{\pi G\rho_1(1+B)^2}\right)^{1/2} \left[1 + \left(\frac{1-B}{1+B}\right)\cos\theta\right]^{-1} \\ &= \frac{a(1-\epsilon^2)}{1+\epsilon\cos\theta}, \end{aligned} \quad (24)$$

This is the equation of an ellipse with eccentricity, ϵ, and semi-major axis, a, with the origin located at one of the focii. The three-dimensional iso-density surfaces are the surfaces of revolution about the major axis. Note that the case $B = 1$ corresponds to the singular isothermal sphere.

Next, we demonstrate that the solutions with $B \neq 1$ are not force-free. Let us consider an equidensity (which is also an equipotential) surface with density ρ_1. The gravitational potential inside the surface produced by the "outside" mass, $\rho < \rho_1$, is linear in the vertical coordinate, z, as shown in Appendix C:

$$\begin{aligned} \Phi = -z\,\sqrt{2\pi G c_s^2}\,\sqrt{\rho_1}\left(B^{1/2} - B^{-1/2}\right) \\ \times\,\frac{1-\epsilon^2}{\epsilon^2}\left[\frac{1}{2|\epsilon|}\ln\left(\frac{1+|\epsilon|}{1-|\epsilon|}\right) - 1\right]. \end{aligned} \quad (25)$$

As one can see, $\Phi \propto \sqrt{\rho_1}$. On the other hand, the mass inside the equipotential is $M = \pi g^{3/2}/\sqrt{\rho_1}\left(B^{1/2} + B^{-1/2}\right)$, i.e., $M \propto 1/\sqrt{\rho_1}$, as follows from Eq. (22). The total gravitational force exerted by the "outside" mass on the "inside" mass, $\mathbf{F}_g = -M\,\nabla\Phi$, is, thus, independent of ρ_1, i.e., independent of the the equidensity surface chosen:

$$\mathbf{F}_g = -\hat{z}\,\frac{2c_s^4}{G}\,\frac{1}{\epsilon}\left[\frac{1}{2|\epsilon|}\ln\left(\frac{1+|\epsilon|}{1-|\epsilon|}\right) - 1\right], \quad (26)$$

where $\epsilon = -(B-1)/(B+1)$. Taking an equipotential surface infinitely close to the origin, one can see that there is a net finite gravitational pull, $\mathbf{F}_g$, (upwards for $B > 1$ and downwards for $B < 1$) acting on the matter at the singularity, $r = 0$. This force is produced by the gravity of all the mass of the system. For the system to remain in equilibrium, there must be an equal and opposite external force, $-\mathbf{F}_g$, applied at $r = 0$. A similar consideration shows that there must also be an extra force acting at infinity or at the last equidensity surface of a finite system. These external forces are uniquely related to the asymmetry parameter, B, and, hence, determine the structure of the equilibrium. When $B = 1$, the external forces vanish and such an equilibrium configuration is force-free. While this analysis is restricted to the non-rotating case, $A = 2$, similar results should hold for any A. We have verified this numerically.

As we mentioned earlier, the asymmetric solutions apparently contradict Lichtenstein's theorem (Lichtenstein 1933; Wavre 1932), according to which any rotating body for which the angular velocity does not depend on z (which is true for a self-gravitating isothermal system) always possesses an equatorial plane of symmetry which is perpendicular to the axis of rotation. There are two assumptions (among others) used in the theorem which are violated by our $B \neq 1$ solutions: (i) no exernal potential is allowed and (ii) the system is assumed to be bounded and the density is nowhere infinite. As we have demonstrated, an asymmetric configuration with $B \neq 1$ exists only when external forces act on the system and pull it apart, in violation of (i). These forces are applied at the positions where condition (ii) is violated: either ρ or r is infinite, and the solution is ill-defined.

7. THE THIN DISK LIMIT, $A \to \infty$

The solutions with $A > 2$ have rotation. The systems become flattened as the rotation increases and they tend to a thin disk as $A \simeq v_{\varphi 0}^2/c_s^2 \to \infty$. In these solutions the parameter B determines the "inclination angle." We define the inclination angle (or the opening angle of the "equatorial cone"), θ_m, as the angle at which the density is maximum for a fixed radial distance, r. In the large-A limit, we can neglect the $\sin^2\theta$ in the denominator in Eq. (14a). We then find

$$\theta_m \simeq 2\,\mathrm{arc\,cot}\left(B^{1/A}\right). \quad (27)$$

Taking into account that $\mathrm{arc\,cot}(x) = \pi/2 - \mathrm{arc\,cot}(1/x)$ and expanding the solution (14a) about θ_m, for small angles $\theta - \theta_m \equiv \vartheta \ll \mathrm{arc\,cot}\left(B^{\pm 1/A}\right)$, we obtain

$$\rho(r,\vartheta) \simeq \frac{c_s^2}{8\pi G}\,\frac{A^2}{r^2}\left(\frac{1+B^2}{2B}\right)^2 \mathrm{sech}^2\left[\vartheta\,A\left(\frac{1+B^2}{2B}\right)\right], \quad (28)$$

For large A, the sech^2 function is very sharply peaked. Thus, the height-to-radius ratio of the disk may be estimated as

$$\frac{H}{R} \simeq \Delta\vartheta \simeq \frac{c_s^2}{v_{\varphi 0}^2}\left(\frac{2}{B + 1/B}\right). \quad (29)$$

In the symmetric case, $B = 1$, this reduces to the standard result, $H/R \sim c_s^2/v_{\varphi 0}^2 \ll 1$. As $c_s \to 0$ the disk becomes the cold Mestel disk having a flat rotation profile.

Toomre (1999, private communication) has studied the dynamics of a truncated cold thin "conical" disk with $A \to \infty$, $B \neq 1$. Figure 6 shows his results for the evolution of a conical disk which is at $t = 0$ has a self-similar form calculated in this paper. No external forces are applied at either the inner edge or outer edge of the disk. Because of the unbalanced gravitational force acting on the innermost ring of the disk, the core region is accelerated upward. The system evolves as a function of time in a self-similar fashion, as shown by a sequence of curves in Figure 6 which corresponds to equally spaced times in $\log t$. The initial conical disk is thus destroyed within a dynamical time, showing that a tree conical disk is highly unstable. On the other hand, if a downward external force of appropriate magnitude is applied on the innermost ring of the conical disk and an equal upward force is applied on the outermost ring, the system is fully stable.

8. EQUILIBRIUM OF COLLISIONLESS STELLAR SYSTEMS

There is a close analogy between fluid (gaseous) self-gravitating equilibrium objects and stellar collisionless systems (cf., Binney & Tremaine 1987). The analogy arises because a fluid system is supported against gravity by pressure gradients, while a stellar system is supported by gradients in the stress tensor which in many respects are like the pressure except that they can be anisotropic. There is a unique connection between the density and velocity fields of fluid equilibrium configurations and the distribution functions of stars in their collisionless analogs. For any stellar distribution function, f, the profiles of density, streaming velocity, $\bar{v}_\varphi$, velocity dispersion, etc., are calculated uniquely by taking the moments of f. Similarly, given ρ, $\bar{v}_\varphi$, etc., an equilibrium $f(\mathbf{x}, \mathbf{v})$ may be determined. In general, there is no guarantee that the stellar distribution function so obtained will be physical, i.e., nonnegative over the entire six-dimensional phase space (see, e.g., Binney & Tremaine 1987 for more discussion). Remarkably, however, the distribution finction is guaranteed to be positive for isothermal systems: the structure of an equilibrium of a self-gravitating isothermal gas is identical to the structure of a collisionless system of stars. We now determine the distribution function of a stellar system which corresponds to the hydrostatic equilibria found in §4.

We consider here the simple case when the distribution function depends on two classical integrals of motion, the energy and axial component of the angular momentum.[4] Lynden-Bell (1962), and subsequently Hunter (1975), developed a method to derive distribution functions from the density and mean velocity profiles. We introduce the integrals of energy and angular momentum of a particle as follows,

$$\mathcal{E} = \left(v_r^2 + v_\theta^2 + v_\varphi^2\right)/2 + \phi(r,\theta) - \phi_0, \qquad \mathcal{L}_z = |v_\varphi|\; r\sin\theta, \tag{30}$$

where v_i are the components of the particle velocity. Since the density profile is insensitive to the direction of rotation, it yields, upon inversion, a distribution function which is even in $\mathcal{L}_z$, i.e., $f_+ = f(\mathcal{E}, \mathcal{L}_z) + f(\mathcal{E}, -\mathcal{L}_z)$. Thus, there are, in principle, infinitely many distributions which produce identical density profiles and differ by an arbitrary, odd in $\mathcal{L}_z$, function, $f_- = f(\mathcal{E}, \mathcal{L}_z) - f(\mathcal{E}, -\mathcal{L}_z)$, determined by the sense of motion of individual stars. For real stellar systems, f_- can also be determined, given the average rotation profile, $\bar{v}_\varphi$. Henceforth, we focus on the symmetric part of the distribution function. We omit the subscript "+," since this should not cause any confusion.

To determine f, we need to express the density profile as a function of the cylindrical radius and gravitational potential. From Eqs. (14b) and (15), it follows that

$$\rho(R,\phi) = R^{A-2}\exp\left[-\left(\phi - \phi_0\right)/c_s^2\right]. \tag{31}$$

We now observe that $\rho(R,\phi)$ is independent of B. Hence $f(\mathcal{E}, \mathcal{L}_z)$ derived from it is also independent of B.[5] Since $f(\mathcal{E}, \mathcal{L}_z)$ is unique (up to f_-), the distribution function proposed by Toomre (1982) as the starting point of his analysis for the specific case of $B = 1$, is, in fact, valid for any value of B and reads,

$$f(\mathcal{E}, \mathcal{L}_z) = f_0\, \mathcal{L}_z^{A-2}\exp\left(-\mathcal{E}/c_s^2\right), \tag{32}$$

where f_0 is a normalization constant. We can now write the phase-space distribution function of a steady-state, self-gravitating system of stars, $f(\mathbf{x}, \mathbf{v})$, in an explicit form as follows,

$$f(r,\theta,\mathbf{v}) = f_0\,\rho(r,\theta)\,\exp\left(-\frac{v_r^2 + v_\theta^2 + v_\varphi^2}{2\,c_s^2} + \frac{v_{\varphi 0}^2}{c_s^2}\,\ln|v_\varphi|\right). \tag{33}$$

Here $c_s^2 \equiv \overline{v^2} - \overline{v}^2$ is the velocity dispersion of stars and $|v_{\varphi 0}|$ is their mean circular velocity. We again notice a remarkable fact: the parameter B defines the shape of the gravitational potential, but does not affect the velocity distribution of stars in this potential.

9. CONCLUSION

In this paper, we analytically calculated all possible self-similar, axisymmetric equilibrium states of a self-gravitating, isothermal gas with rotation. We showed that there are two distinct classes of hydrostatic equilibria; namely cylindrically symmetric equilibria and axially symmetric equilibria. The axially symmetric solutions are more physical, since the matter density vanishes at infinity. Among the axially symmetric solutions, we found equilibrium states which are asymmetric with respect to the equatorial plane. Such states satisfy Poisson's equation, but are not force-free at the singularities, $r = 0, \infty$. It is the external forces that support the asymmetric shape. This example shows that self-similar solutions should be treated with caution and checked for possibly unphysical boundary conditions that may be "hidden" at their singular points.

[4]According to Jeans' theorem, the distribution function of a steady state, self-gravitating system may be presumed to be a function of at most three isolating integrals. Usually, two of the integrals are the energy and axial angular momentum, while, in general, there is no simple analytical form for the third integral (see, Binney & Tremaine 1987 for discussion).

[5]More precisely, it depends on B via $\phi(r,\theta)$ entering the definition of $\mathcal{E}$.

Are the asymmetric configurations likely to be realized in nature? Since real systems are finite, we should truncate our solutions at some ρ_c and throw away the "outside" mass. For a truncated solution to be valid, the gravitational potential of the discarded "outside" mass has to be replaced by a suitable external potential. As Eq. (25) shows, the potential must have a linear gradient in z (for a nonrotating solution) of a magnitude determined by ρ_c and B. Such a potential may be produced, for instance, by an external object located on the z axis at a distance large compared to the size of the system. Second, to keep the system at rest, another force of equal magnitude and opposite sign must act on the central core. Since the force must act on the core alone, it must be of non-gravitational origin, which, clearly, is not easy to arrange. One could imagine, for example, that the core is ionized (by radiation of a central source, for instance), while the material outside the core is neutral. Then, if an external magnetic field threads the ionized core, one could imagine the field pinning the core, but having no influence on the neutral outer material. The distortion from equatorial symmetry would then be determined by the strength of the gravitational attraction of the external object and the counterbalancing force from the tension and curvature of the field lines. Although this construction is rather artificial, it demonstrates that asymmetric, non-force-free equilibria may, at least in principle, exist in nature. If they do, and if the systems are isothermal, they will resemble some of the $B \neq 1$ solutions derived in this paper.

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APPENDIX

METHOD OF SOLUTION OF EQ. (13)

Here we demonstrate how a general solution of Eq. (13) can be found analytically. The equation we analyze reads

$$2 - \frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d}{d\theta}\,\ln f(\theta)\right) = \kappa f(\theta), \tag{A1}$$

where $\kappa = 4\pi G\rho_0/c_s^2$. First of all, we get rid of the constant term on the left-hand-side. We observe that

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d}{d\theta}\,\ln(\sin^\beta\theta)\right) = -\beta, \tag{A2}$$

where β is a constant. Thus, upon introducing a new function $g(\theta) = f(\theta)/\sin^2\theta$, equation (A1) becomes

$$\sin\theta\frac{d}{d\theta}\left(\sin\theta\frac{d}{d\theta}\,\ln g(\theta)\right) + \kappa g(\theta) = 0. \tag{A3}$$

We now observe that $\sin\theta$ can be put into the denominator as follows, $\partial\theta/\sin\theta = \partial\ln|\tan(\theta/2)|$. Let us redefine again the function $g(\theta)$ and change the independent variable θ as follows,

$$\xi = \ln\left|\tan\frac{\theta}{2}\right|, \qquad w(\xi) = \ln g(\theta) = \ln\left(\frac{f(\theta)}{\sin^2\theta}\right), \tag{A4}$$

Then the differential equation is greatly simplified and reads[6]

$$\frac{d^2}{d\xi^2}\,w(\xi) + \kappa\,e^{w(\xi)} = 0. \tag{A5}$$

One can now reduce this equation to a first-order differential equation which can be integrated. Since Eq. (A5) contains no explicit dependence on ξ, we introduce a *new* function p which is a function of the *old* function w as follows,

$$p(w) = \frac{dw(\xi)}{d\xi}, \text{ so that } \frac{d^2w(\xi)}{d\xi^2} = \frac{dp(w)}{d\xi} = \frac{dp(w)}{dw}\frac{dw(\xi)}{d\xi} = p(w)\frac{dp(w)}{dw}. \tag{A6}$$

Upon this substitution, the resulting differential equation, $p\,p'_w = -\kappa\,e^w$, is readily integrated for $p(w)$ to yield

$$\frac{dw(\xi)}{d\xi} \equiv p(w) = \pm\sqrt{C_1 - 2\kappa\,e^{w(\xi)}}. \tag{A7}$$

[6]Note that the above substitutions are equivalent to Howard's transformation used by Toomre (1982).

Here C_1 is a first constant of integration which we assume, to be specific at the moment, to be positive (see discussion below). Making use of another substitution,

$$v(\xi) = e^{w(\xi)}, \tag{A8}$$

the resulting differential equation, $v'_\xi = \pm v\sqrt{C_1 - 2\kappa v}$, can be integrated again. We arrive at the expression:

$$\frac{1}{\sqrt{C_1}} \ln\left|\frac{\sqrt{C_1 - 2\kappa v(\xi)} - \sqrt{C_1}}{\sqrt{C_1 - 2\kappa v(\xi)} + \sqrt{C_1}}\right| = \pm(\xi - C_2), \tag{A9}$$

where C_2 is a second constant of integration. We now resolve this equation with respect to $v(\xi)$. There are two solutions, namely

$$v(\xi) = \pm\frac{2C_1}{\kappa}\frac{e_*(1 \mp e_*)^2}{(1 - e_*^2)^2}, \tag{A10}$$

where we used the short-hand notation $e_* = \exp\left(\pm\sqrt{C_1}\,[\xi - C_2]\right)$. Comparing Eqs. (A4) and (A8), one can see that the density $\rho(r,\theta) \propto f(\theta) \propto v(\xi)$. Thus, for the density to be positive, the function $v(\xi)$ must be *real* and *positive*, also. This condition is satisfied for the upper sign in Eq. (A10). Thus, we have

$$v(\xi) = \frac{2C_1}{\kappa}\frac{e_*}{(1 + e_*)^2} = \frac{C_1/2\kappa}{\cosh^2\left(\sqrt{C_1}\,[\xi - C_2]/2\right)}. \tag{A11}$$

Recalling all the definitions we made throughout the calculation, see Eqs. (A4) and (A8), we finally obtain the general solution of the differential equation (13),

$$f(\theta) = \frac{C_1/\left(2\kappa \sin^2\theta\right)}{\cosh^2\left(\sqrt{C_1}\,[\ln|\tan(\theta/2)| - C_2]/2\right)}. \tag{A12}$$

Clearly, since $\cosh x$ has no zeros for $-\infty < x < \infty$, this solution is well behaved for all θ's from the range: $0 < \theta < \pi$. Therefore, we investigate some properties of this solution in more details in the main text. Quite interestingly, solution (A12) can be simplified further and finally reads as follows,

$$f(\theta) = \frac{2A^2}{\kappa \sin^2\theta}\frac{B\,|\tan(\theta/2)|^A}{\left(1 + B\,|\tan(\theta/2)|^A\right)^2}, \tag{A13}$$

where $A = \sqrt{C_1}$ and $B = \exp\left(-\sqrt{C_1}C_2\right) > 0$ are new constants of integration. The "absolute value" signs may be omitted if $0 \le \theta \le \pi$, i.e., the tangent is non-negative.

It can be rigorously shown that the particular choice of $C_1 > 0$ and C_2 being a real number is unique, provided the solution to be physically meaningful. The derivation following Eq. (A7) holds for the complex-valued constants C_1 and C_2. Separating real and imaginary parts in Eq. (A10) and requiring v to be a *real-valued* function for *all* real ξ, one can arrive, upon straightforward but cumbersome mathematical manipulations, at the conclusion that C_1 must be positive and C_2 may be complex, such that $\mathrm{Im}\,\sqrt{C_1}C_2 = \pi k,\ k = 0, 1, 2, \ldots$ and its real part is arbitrary. From this, it follows that $e_* > 0$ for even k's and $e_* < 0$ for odd k's. The additional constraint that v must be positive uniquely selects the solution given by Eq. (A11).

EXPRESSIONS FOR THE MASS, GRAVITATIONAL POTENTIAL ENERGY, AND ANGULAR MOMENTUM

From Eqs. (14), using Eqs. (15) and (19), we straightforwardly calculate

$$M = \frac{2\pi g^{3/2}}{\sqrt{\rho_c}}\,\mathcal{I}, \tag{B1a}$$

$$W = -c_s^2\,\frac{\pi g^{3/2}}{\sqrt{\rho_c}}\left\{\left[\ln\rho_c + 2 - \frac{\phi_0}{c_s^2} - (A - 2)\left(\ln\sqrt{g/\rho_c} - 1\right)\right]\mathcal{I} - \left(\frac{A}{2} - 1\right)\mathcal{J}\right\}, \tag{B1b}$$

$$L = \frac{\pi g^2}{\rho_c}\,v_{\varphi 0}\,\mathcal{K}, \tag{B1c}$$

where $g = c_s^2/(2\pi G)$ and $\mathcal{I}$, $\mathcal{J}$, and $\mathcal{K}$ are the integrals:

$$\mathcal{I} = \int_0^\pi [\Theta(\theta)]^{3/2}\sin\theta\,\mathrm{d}\theta, \qquad \mathcal{J} = \int_0^\pi [\Theta(\theta)]^{3/2}\,\ln\left[\Theta(\theta)\sin^2\theta\right]\sin\theta\,\mathrm{d}\theta, \qquad \mathcal{K} = \int_0^\pi [\Theta(\theta)]^2\,\sin^2\theta\,\mathrm{d}\theta. \tag{B2}$$

Changing the integration variable to $z = B\tan^A(\theta/2)$, the above integrals reduce to

$$\mathcal{I} = \frac{A^2}{2}\int_0^\infty \left(B^{1/A}z^{1/2-1/A} + B^{-1/A}z^{1/2+1/A}\right)\frac{1}{(1+z)^3}\,\mathrm{d}z, \tag{B3a}$$

$$\mathcal{J} = \frac{A^2}{2}\int_0^\infty \left(B^{1/A}z^{1/2-1/A} + B^{-1/A}z^{1/2+1/A}\right)\frac{1}{(1+z)^3}\,\ln\left(\frac{A^2 z}{(1+z)^2}\right)\,\mathrm{d}z, \tag{B3b}$$

$$\mathcal{K} = \frac{A^3}{2}\int_0^\infty \left(B^{1/A}z^{1-1/A} + B^{-1/A}z^{1+1/A}\right)\frac{1}{(1+z)^4}\,\mathrm{d}z. \tag{B3c}$$

The first integral, $\mathcal{I}$, is symbolically identical to $I = \int_0^\infty z^{\alpha-1}R(z)\,\mathrm{d}z$, where $R(z)$ is a rational function. It can be evaluated in the complex plane. The branch cut is taken to be from 0 to $+\infty$ along the real axis. We choose the contour of integration which consists of four pieces: (i) a circular part around the branching point $z = 0$ of radius $R_< \to 0$, (ii) a linear part, L_+, along the cut from above extending from $R_<$ to $R_>$, (iii) a circular part of radius $R_> \to \infty$, and (iv) a linear part, L_-, going along the cut from below and closing the contour. One can see that the integrals over the circular pieces vanish when appropriate limits are taken, the integral along L_+ is the integral we are looking for, and the integral along L_- differ from it by a phase shift. Then, it becomes

$$I = \int_0^\infty z^{\alpha-1}R(z)\,\mathrm{d}z = \frac{2\pi i}{1-e^{2\pi i\alpha}}\sum \mathrm{res}_{z_k}\left(z^{\alpha-1}R(z)\right), \tag{B4}$$

where res_{z_k} denotes the residue of a function at a pole z_k and sum goes over all poles of function $R(z)$. Applying this method, we obtain

$$\mathcal{I} = \frac{(A^2-4)}{16}\frac{\pi}{\cos(\pi/A)}\left(B^{1/A} + B^{-1/A}\right). \tag{B5}$$

The second integral, $\mathcal{J}$, consists of three terms, due to the logarithm of a product. The first of them is identical to I, i.e., $J_1 = \int_0^\infty z^{\alpha-1}R(z)\ln A^2\,\mathrm{d}z = I\ln A^2$. The second one, $J_2 = \int_0^\infty z^{\alpha-1}R(z)\ln z\,\mathrm{d}z$, may be evaluated either directly via residues, as above, because $\ln z$ does not affect the convergence of the integral, or by noticing that $\int_0^\infty z^{\alpha-1}R(z)\ln z\,\mathrm{d}z = \frac{\partial}{\partial\alpha}\int_0^\infty z^{\alpha-1}R(z)\,\mathrm{d}z$ and using the previous result. The evaluation of the last integral, $J_3 = \int_0^\infty z^{\alpha-1}(1+z)^{-3}\ln(1+z)\,\mathrm{d}z$, is trickier because there is another branching point, $z = -1$, which coincide with the pole of the rational function. We modify it as follows,

$$\begin{aligned} J_3 &= \int_0^\infty \frac{z^{\alpha-1}\ln(1+z)}{(1+z)^3}\,\mathrm{d}z = -\frac{\partial}{\partial\beta}\int_0^\infty \frac{z^{\alpha-1}}{(1+z)^\beta}\,\mathrm{d}z\bigg|_{\beta=3} \\ &= -\frac{\partial}{\partial\beta}B(\alpha,\beta-\alpha)\bigg|_{\beta=3} = B(\alpha,3-\alpha)\left[\psi(3)-\psi(3-\alpha)\right], \end{aligned} \tag{B6}$$

where $B(m,n) = \Gamma(m)\Gamma(n)/\Gamma(m+n)$ is the beta-function, $\psi(x) = \Gamma'(x)/\Gamma(x)$ is the digamma function, and $\Gamma(x)$ is the gamma-function. Using the identity $\psi(x+1) = \psi(x) + 1/x$ to express $\psi(3)$ and combining all terms together, we arrive at the following result,

$$\begin{aligned} \mathcal{J} = 2\mathcal{I}\left(\ln A - \frac{3}{2} + \gamma\right) &+ \frac{\pi}{16\cos(\pi/A)}\left[8A - \left(A^2-4\right)\pi\tan(\pi/A)\right]\left(B^{1/A} - B^{-1/A}\right) \\ &+ \frac{\pi(A^2-4)}{8\cos(\pi/A)}\left[B^{1/A}\,\psi\left(\frac{3}{2}+\frac{1}{A}\right) + B^{-1/A}\,\psi\left(\frac{3}{2}-\frac{1}{A}\right)\right], \end{aligned} \tag{B7}$$

where $\gamma \equiv -\psi(1) = 0.57721\ldots$ is Euler's constant. This equation may be further simplified using the identities: $\psi(x+1) = \psi(x) + 1/x$ and $\psi\left(\frac{1}{2}+x\right) - \psi\left(\frac{1}{2}-x\right) = \pi\tan(\pi x)$. We arrive at the following result,

$$\mathcal{J} = \mathcal{I}\left[\ln A^2 - 3 + 2\gamma + \frac{4A^2}{A^2-4} - \pi\tan\frac{\pi}{A} + 2\,\psi\left(\frac{1}{2}+\frac{1}{A}\right)\right]. \tag{B8}$$

The last integral, $\mathcal{K}$, is similar to $\mathcal{I}$. It is evaluated via the residue theorem to yield

$$\mathcal{K} = \frac{\pi(A^2-1)}{12\sin(\pi/A)}\left(B^{1/A} + B^{-1/A}\right). \tag{B9}$$

GRAVITATIONAL POTENTIAL OF A TRUNCATED, NONROTATING SYSTEM

Here we analytically calculate the gravitational potential of the nonrotating, $A = 2$, system truncated along the cutoff equidensity (and, hence, equipotential) surface with density $\rho = \rho_c$. Note, since we cut the system along the equipotential surface, the potential produced by the "inside" mass in the outer space and the potential inside the cavity due to all "outside" mass are of equal magnitude and add up to a constant. The direct evaluation of the gravitational potential via the volume integration involves elliptic integrals and turns out to be cumbersome. We use a different approach.

Let us assume, without loss of generality, that $B > 1$. Then, the ellipsoids are shifted upwards, as follows from Eq. (23) and plotted in Fig. 7. We label each ellipsoid with the quantity

$$\Delta = a\,\epsilon = \Delta_0/\sqrt{\rho}, \tag{C1}$$

where a is the major axis, ϵ is the eccentricity, and $\Delta_0 = -\sqrt{c_s^2/8\pi G}\,\left(B^{1/2} - B^{-1/2}\right)$. We now consider two such confocal ellipsoids; the quantities referred to a larger one are denoted by the "prime", as shown in Fig. 7. Let us now fill the space between the two ellipsoids with matter of a homogeneous density ρ. The gravitational potential in the empty space *inside* the smaller ellipsoid is equal to the potential inside the large homogeneous ellipsoid, $\Phi'_{\rm int}(\mathbf{x})$, less the potential inside the small one, $\Phi_{\rm int}(\mathbf{x})$, having the same density. The gravitational potential in the interior of a homogeneous prolate ellipsoid centered at the origin is known (Binney & Tremaine 1987):

$$\Phi_{\rm int}(\mathbf{x}) = -\pi G\rho\left(I\,b^2 - \sum_{i=1}^{3} A_i\,x_i^2\right), \tag{C2}$$

where $b = a\sqrt{1-\epsilon^2}$ is the minor axis, $\mathbf{x} = (x, y, z)$, and

$$I = \frac{\sqrt{1-\epsilon^2}}{\epsilon}\ln\left(\frac{1+\epsilon}{1-\epsilon}\right), \tag{C3a}$$

$$A_1 = A_2 = \frac{1-\epsilon^2}{\epsilon^2}\left[\frac{1}{1-\epsilon^2} - \frac{1}{2\epsilon}\ln\left(\frac{1+\epsilon}{1-\epsilon}\right)\right], \tag{C3b}$$

$$A_3 = 2\,\frac{1-\epsilon^2}{\epsilon^2}\left[\frac{1}{2\epsilon}\ln\left(\frac{1+\epsilon}{1-\epsilon}\right) - 1\right]. \tag{C3c}$$

Taking into account that our coordinate system is shifted through $\Delta < 0$, we write, $\sum A_i x_i^2 = A_1 R^2 + A_3(z+\Delta)^2$. The potential inside the shell bounded by the surfaces Δ' and Δ is, thus,

$$\Phi_{\rm inside} = \Phi'_{\rm int} - \Phi_{\rm int} = -\pi G\rho\left[\left(I\,\frac{1-\epsilon^2}{\epsilon^2} - A_3\right)\left(\Delta'^2 - \Delta^2\right) - 2A_3 z\left(\Delta' - \Delta\right)\right]. \tag{C4}$$

Making Δ' infinitesimally close to Δ, we obtain the contribution to the potential due to the infinitely thin shell of "thickness" $\delta\Delta = \Delta' - \Delta$ and density $\rho = (\Delta_0/\Delta)^2$:

$$\delta\Phi_{\rm inside} = -2\pi G\left(\frac{\Delta_0}{\Delta}\right)^2\left[\left(I\,\frac{1-\epsilon^2}{\epsilon^2} - A_3\right)\Delta\,\delta\Delta - A_3 z\,\delta\Delta\right]. \tag{C5}$$

The gravitational potential inside the truncated system extending from the cutoff surface $\Delta_c = \Delta_0/\sqrt{\rho_c}$ to infinity is simply the integral over all thin shells with $\Delta > \Delta_c$, i.e., $\Phi_{\rm cav} = \int \delta\Phi_{\rm inside}$. The first term in Eq. (C5) evaluates to a constant, which may be absorbed into Φ_0, the zero-level of the potential. The second term yields the coordinate dependent contribution we are looking for,

$$\Phi_{\rm cav}(\mathbf{x}) = -z\,\sqrt{\pi G c_s^2/2}\left(B^{1/2} - B^{-1/2}\right)A_3(\epsilon)\sqrt{\rho_c}, \tag{C6}$$

where $\epsilon = (1-B)/(1+B)$ and $A_3(\epsilon) = A_3(-\epsilon)$ is given by Eq. (C3). We emphasize that the potential is *linear* in the vertical coordinate, z, which corresponds to the homogeneous gravitational field. The strength of this field is determined by B and the cutoff density, ρ_c, only. The gravitational acceleration, $\mathbf{g} = -\nabla\Phi_{\rm cav}$, is upwards for $B > 1$ and downwards for $B < 1$.

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FIG. 1.— Iso-density contours for $A = 0.7$, $B = 1$ and $B = 3$. The horizontal axis corresponds to R and the vertical axis to z.

FIG. 2.— Same as in Fig. 1 for $A = 2$, $B = 1$ and $B = 3$.

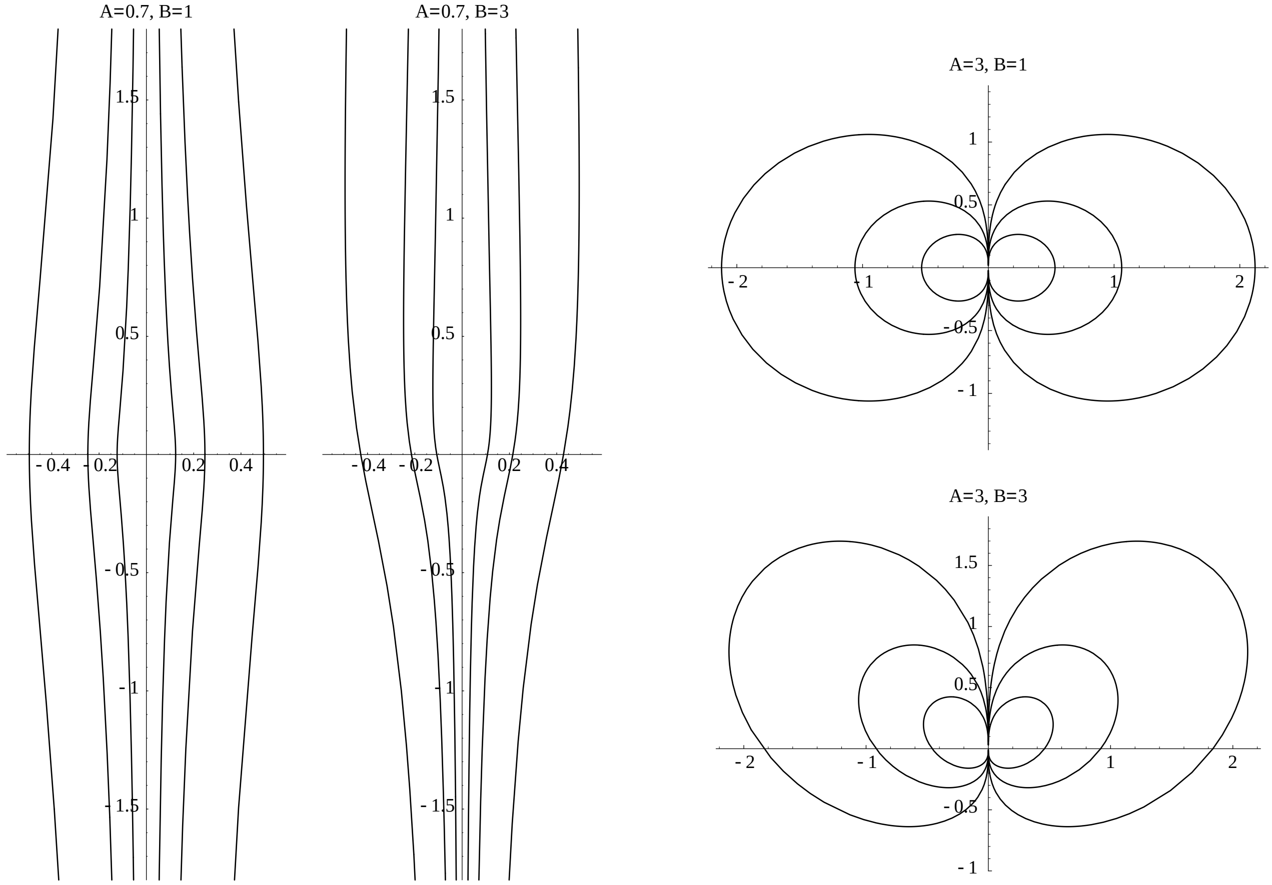
FIG. 3.— Same as in Fig. 1 for $A = 3$, $B = 1$ and $B = 3$.

FIG. 4.— Same as in Fig. 1 for $A = 20$, $B = 1$ and $B = 10^5$.

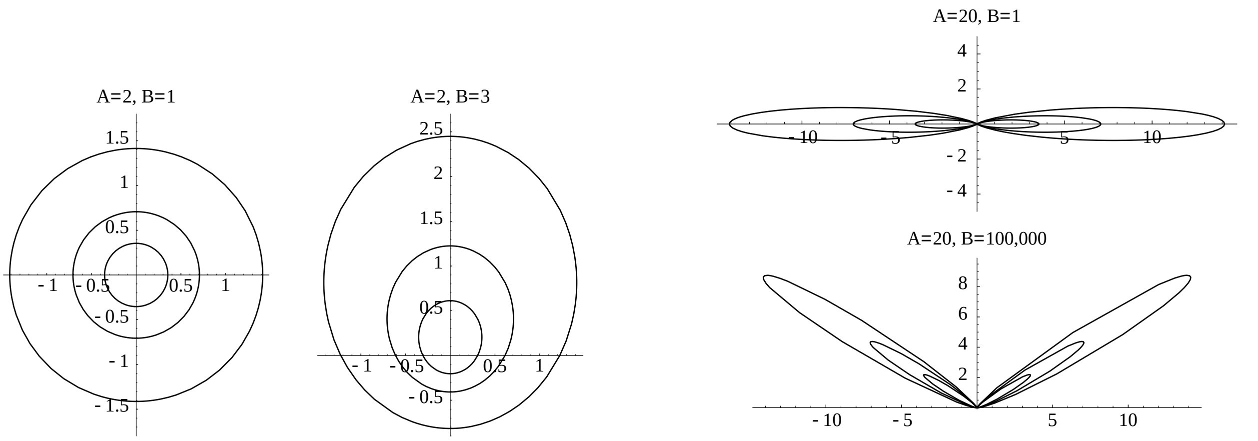
FIG. 5.— The total mass M (solid curve), gravitational energy W (short-dashed curve), and angular momentum L_z (long-dashed curve), of a finite system vs. A. All quantities are in arbitrary units.

FIG. 6.— The evolution of a free axially symmetric "conical disk" as calculated by Toomre (1999, private communication). Radial profiles of density in cylindrical coordinates are shown at times: $t = 0$ (straight line) and (from botton to top) $t = 0.25,\ 0.5,\ 1,\ 2,\ 4$. The coordinates and time are in arbitrary units.

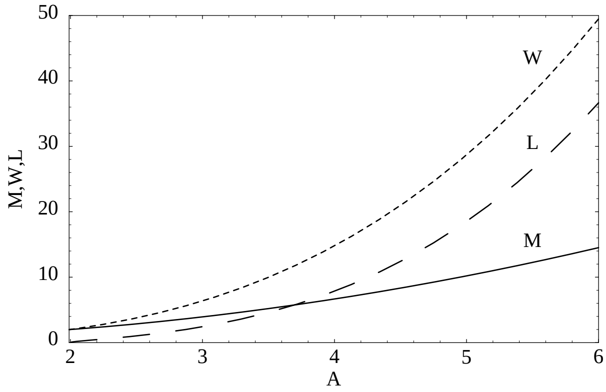
FIG. 7.— Two confocal ellipsoids, used to calculate the potential of a nonrotating truncated system.

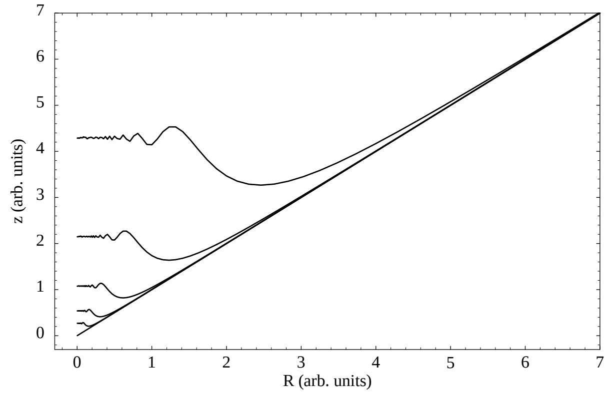


Figs. 1,3



Figs. 2,4


W
L
M
M,W,L
A
50
40
30
20
10
0
2
3
4
5
6


z (arb. units)
R (arb. units)
0
1
2
3
4
5
6
7

Figs. 5,6

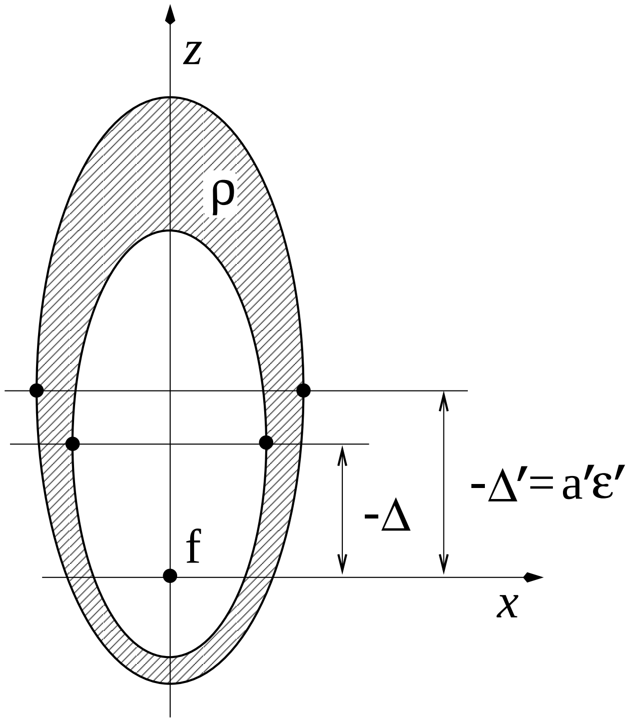

z
ρ
f
-Δ
-Δ′=a′ε′
x

Fig. 7